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Efficient Solutions for Pricing and Hedging Interest Rate Asian Options

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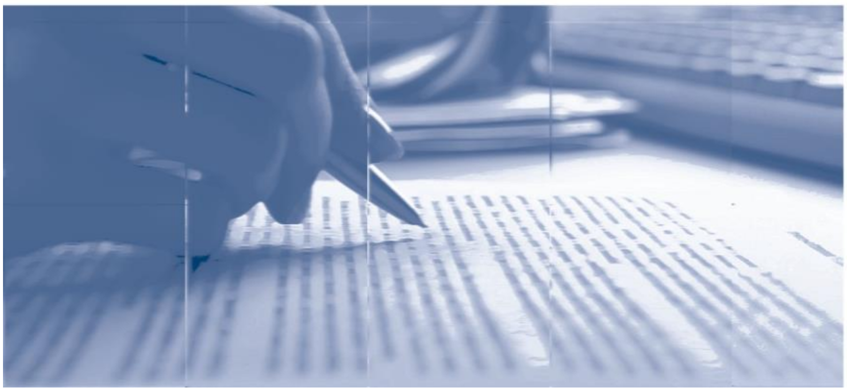

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Non-Technical Summary

Fixed income options are important assets because they allow agents to build speculative or hedging strategies on interest rate. In Brazil, the main fixed-income option is the IDI option that has as underlying the accumulated DI rate between the opening of the position and the maturity. Several papers and studies discuss methodologies and formulas to pricing the IDI option. In complex models, the computational solution to the problem of pricing IDI options is a challenge. In this work, we propose an easy-to-implement numerical technique that can be used to effectively evaluate the IDI options. The method is based on expansions in cosine series. Through several examples, we illustrate the versatility of the method as well as its computational speed.

Sumário Não Técnico

Opções de renda fixa são ativos importantes porque permitem que os agentes especulem ou façam *hedge* de taxa de juros. No Brasil, a principal opção de renda fixa é a opção IDI, que tem como ativo objeto a taxa DI acumulada entre a abertura da posição e o vencimento. Diversos trabalhos e estudos discutem metodologias e fórmulas de apreçamento da opção IDI. Em modelos complexos, a solução computacional do problema de apreçamento de opções IDI é um desafio. Neste trabalho, nós propomos uma técnica numérica de fácil implementação que pode ser usada para apreçar efetivamente as opções IDI. O método é baseado em expansões em séries de cosseno. Por meio de diversos exemplos, nós ilustramos a versatilidade do método bem como a sua rapidez computacional.

Efficient Solutions for Pricing and Hedging Interest Rate Asian Options

Allan Jonathan da Silva*

Jack Baczynski†

José Valentim Machado Vicente‡

Abstract

We develop analytical solutions for the characteristic function of the integrated short-term rate process using the Fourier-cosine series. The method allows us to study the pricing of Asian interest rate options for a broad class of affine jump-diffusion models. In particular, we provide closed-form Fourier-cosine series representations for the price and the delta-hedge of Asian interest rate options under the augmented Vasicek model. In a numerical study, we show that Asian interest rate option prices can be accurately and efficiently approximated by truncating their series representations. The proposed procedure is calculated fast and is superior in accuracy when compared to the existing numerical methods used to price Asian interest rate options.

Keywords ou Palavras-chave: Interest rate derivatives, Fourier Series, Affine jump-diffusion, COS method.
JEL Classification ou Classificação JEL: C02, G12.

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1 Introduction

The interest rate derivatives market differs across countries. The existing singularities in the Brazilian market stirred the development of particular derivatives contracts. Among them, we have a singular kind of interest rate option, namely the Interbank Deposit rate Index (IDI) option. The IDI option is the main interest rate option offered by B3 (the Brazilian stock exchange). It is of European type with cash settled at maturity. The IDI is an index that accumulates from an initial value according to the daily Interbank Deposit rate (DI rate).[1] Note that the IDI is equivalent to a bank account. In a continuous framework, the logarithm of IDI is a proxy for the integrated short-term rate.

Although IDI option is based on the integrated interest rate, we can view the DI rate as its underlying. In this case, the IDI option is an Asian option since its payoff depends on the path of the DI rate during the life of the option. Among other advantages, Asian options reduce the risk of market manipulation of the underlying instrument at maturity. Moreover, the averaging feature makes Asian options typically cheaper than vanilla options. However, the pricing of Asian options is not easy even if we consider simple frameworks such as Gaussian models.

In the last years, some interesting results about the IDI option pricing have been developed. We can find closed-form expression for the IDI option price under the Vasicek model (see Vieira and Pereira, 2000) and the CIR model (see Barbachan and Ornelas, 2003). Junior et al. (2003) and Almeida et al. (2003) assume that the short-term rate follows the Hull and White (1993) model and obtain analytical solutions for the price of IDI options. Barbedo et al. (2010) implement the HJM model to price IDI options. The problem is numerically solved via a finite difference method in da Silva et al. (2016). The authors show significant discrepancies in using the same interest rate model for discretely and continuously compounded framework. An interesting result is provide by Genaro and Avellaneda (2018), where the price of IDI option is sensitive to changes in monetary policy. Baczynski et al. (2017) propose an alternative generic procedure to pricing IDI options. An empirical and economical study regarding IDI options can also be found in Almeida and Vicente (2012).

However, there is a lack in the literature of a general fast and accurate IDI option pricing method, which includes any affine jump-diffusion model. In this paper we partially fill this gap by applying the COS method to the IDI option pricing problem. The COS method is a new Fourier inversion method that has experienced an increase in its usage. It is a procedure to calculate probability density functions and option prices via Fourier-cosine series introduced in Fang and Oosterlee (2008). In several numerical experiments, these authors show that the convergence rate of the COS method is exponen-

[1] The DI rate is an inter-bank one-day interest rate close-related with the Brazilian prime rate (Selic rate). It represents the average rate of inter-bank overnight transactions in Brazil.

tial and its computational complexity is linear. They also present evidences that the COS method improve the speed of pricing plain vanilla and exotic options when compared to traditional procedures.

Many works have recently extended the original COS model and applied it to solve a variety of option pricing problems under classical models. Zhang and Oosterlee (2013) show how to use the COS method to pricing arithmetic and geometric Asian options under exponential Lévy processes. Zhang and Feng (2019) find the price of American put options under the double Heston (1993) model using the COS method. Zhang et al. (2012) analyze the efficiency properties of pricing commodity options with early-exercise. Ballotta et al. (2018) employ the COS method in multivariate structural models to pricing credit default using Lévy processes. Tour et al. (2018) apply the Fourier cosine expansion method to calculate the price of several options under regime-switching models. Have and Oosterlee (2018) use the COS method for option valuation under the Stochastic Alpha Beta Rho (SABR) dynamics. The pricing of forward starting options with stochastic volatility and jumps is considered in Zhang and Geng (2017). Crisóstomo (2018) compare the CPU effort of the COS method against six other Fourier-based schemes and concluded that it is notably the fastest.

Although the vast existing literature cited above and others not mentioned, to the best of our knowledge there is no application of the COS method to interest rate derivatives pricing. In this paper, we accomplish an original contribution to the pricing of Asian interest rate options by applying the COS method to the IDI option. Besides describing the procedure to find prices of Asian options, we provide delta-neutral strategies to hedge the accumulated interest rate. In a simple numerical exercise, we assume a Gaussian process to the short-term rate and compare the solution obtained by the COS method to the closed-form expression given in Vieira and Pereira (2000). We find that the COS method generates fast, accurate and stable prices.

In order to highlight the versatility of the COS method, we also study the pricing and hedging of a digital option underlain on the IDI. Although this kind of option is not currently offered by B3, this exercise shows that we can easily extend the class of payoffs to encompass exotic options whose pricing, using traditional techniques, is surrounded by more difficult numerical issues.

The paper is organized as follows. In Section 2 we review the Fourier-cosine expansion method to recover density functions and calculate the price of financial derivatives. In Section 3 we present the pricing problem and show some exponential affine characteristic functions of integrated stochastic interest rate processes aimed to price derivatives of Asian type with the COS method. We also present the cumulants of the random variables and recover the associated probability density function with the series representation derived from the Fourier Transforms. In Section 4 we show how the COS method can price

IDI options. Hedging parameters is considered in Section 5. In Section 6 we analyze the computational issues of the method and conclude the article in Section 7.

2 Fourier series method

Let x_t be the price at time t of the underlying of a European call option maturing in t. Denote by f the probability density function of x_t. Thus, the price of this option at time 0 is

$$\begin{aligned} C(0,t) &= \mathbb{E}\left[g(x_t)\right] \\ &= \int_{\mathbb{R}} g(x)f(x)\mathrm{d}x, \end{aligned} \tag{1}$$

where g is the discounted payoff function of the option and $\mathbb{E}$ is the risk-neutral expected value. Truncating f in the interval $[a,b]$ we have:

$$C(0,t) \approx \int_a^b g(x)f(x)\mathrm{d}x. \tag{2}$$

The integral in (2) can be calculated by the COS method proposed by Fang and Oosterlee (2008). The COS method is an interesting, fast and accurate derivatives pricing method based on Fourier-cosine series. In what follows, we present the COS method and show how to use it to pricing options.

Let

$$f : [0,\pi] \longrightarrow \mathbb{R}$$

be an integrable function. Then the Fourier-cosine series of f is defined by

$$\frac{a_0}{2} + \sum_{j=1}^{\infty} a_j \cos(j\xi), \quad \xi \in [0,\pi] \tag{3}$$

where

$$a_j = \frac{2}{\pi}\int_0^{\pi} f(\xi)\cos(j\xi)\mathrm{d}\xi, \quad j \geq 0. \tag{4}$$

$$\tag{5}$$

For functions supported in any arbitrary interval $[a,b]$, a change of variable $\xi = \pi\frac{x-a}{b-a}$ is considered. Then, the Fourier-cosine series expansion of f, now defined in the interval

$[a,b]$ is

$$f(x) = \frac{a_0}{2} + \sum_{j=1}^{\infty} a_j \cos\left(j\pi \frac{x-a}{b-a} \right), \tag{6}$$

where

$$a_j = \frac{2}{b-a} \int_a^b f(x) \cos\left(j\pi \frac{x-a}{b-a} \right) \mathrm{d}x, \quad j \geq 0. \tag{7}$$

Let us assume that $f \in L^1(\mathbb{R})$. The approximation in the interval $[a,b]$ of the coefficients of the Fourier-cosine expansion of f is

$$\begin{aligned} a_j &= \frac{2}{b-a} \int_a^b f(x) \Re\left(\mathrm{e}^{ij\pi \frac{x-a}{b-a}} \right) \mathrm{d}x \\ &= \frac{2}{b-a} \Re\left(\mathrm{e}^{-ij\pi \frac{a}{b-a}} \int_a^b f(x) \mathrm{e}^{ij\pi \frac{x}{b-a}} \mathrm{d}x \right). \end{aligned}$$

If f is the probability density function in (2), then

$$a_j \approx \frac{2}{b-a} \Re\left(\mathrm{e}^{-ij\pi \frac{a}{b-a}} \hat{f}\left(\frac{j\pi}{b-a} \right) \right) \triangleq A_j, \tag{8}$$

where $\hat{f}$ is the characteristic function of x_t, that is

$$\hat{f}(u) = \int_{\mathbb{R}} \mathrm{e}^{ixu} f(x) \mathrm{d}x, \tag{9}$$

which can be approximated in a truncated interval by

$$\hat{f}(u) \approx \int_a^b \mathrm{e}^{ixu} f(x) \mathrm{d}x. \tag{10}$$

Therefore, the approximation of f is given by the following Fourier-cosine series

$$f(x) \approx \frac{A_0}{2} + \sum_{j=1}^{n} A_j \cos\left(j\pi \frac{x-a}{b-a} \right), \quad x \in [a,b], \tag{11}$$

for an given n. Therefore,

$$C(0,t) \approx \frac{A_0}{2} \int_a^b g(x) \mathrm{d}x + \sum_{j=1}^{n} A_j \int_a^b g(x) \cos\left(j\pi \frac{x-a}{b-a} \right) \mathrm{d}x. \tag{12}$$

Hence, the series approximation of the option price is given by

$$C(0,t) = \frac{A_0 B_0}{2} + \sum_{j=1}^{n} A_j B_j, \tag{13}$$

where the A_k coefficients are given by (8) and

$$B_j = \int_a^b g(x) \cos\left(j\pi \frac{x-a}{b-a} \right) \mathrm{d}x, \quad \text{for } j = 0, 1, ..., n. \tag{14}$$

The choices of the integration limits for the approximation were proposed in Fang and Oosterlee (2008) as follows:

$$a = c_1 - L\sqrt{c_2 + \sqrt{c_4}} \qquad b = c_1 + L\sqrt{c_2 + \sqrt{c_4}} \tag{15}$$

with $L = 10$. The coefficients c_k are the k-th cumulant of x_t given by

$$c_n = \frac{1}{i^k} \frac{\mathrm{d}^k}{\mathrm{d}u^k} h(u)|_{u=0} \qquad h(u) = \ln \mathbb{E}\left[\mathrm{e}^{iux_t} \right]. \tag{16}$$

3 Problem statement and interest rate models

We assume an interest rate market with underlying probability space $(\Omega, \mathbb{F}, \mathbb{P})$ equipped with a filtration $\mathbb{F} = (\mathscr{F}_t)_{t\in[0,T]}$ where $\mathbb{P}$ is the risk neutral measure. Let r_t be the instantaneous continuously compounding interest rate. We assume that r_t follows a jump-diffusion model given by

$$dr_t \;=\; \mu(r_t, t)dt + \sigma(r_t, t)dB_t + ZdN(\lambda t), \tag{17}$$

where $\mu(r_t, t)$ is the mean, $\sigma(r_t, t)$ is the volatility and B_t is the standard Wiener processes. N is a pure jump process with constant positive intensity λ and jump amplitudes Z, which are i.i.d. and independent of B_t.

According to the B3 protocols, the DI rate is the average of the interbank rate of a one-day-period, calculated daily and expressed as the effective rate per annum.[2] So, the ID index (IDI) accumulates discretely, according to

$$y_t = y_0 \prod_{j=1}^{t-1} (1 + DI_j)^{\frac{1}{252}}, \tag{18}$$

where j denotes the end of day and DI_j assigns the corresponding DI rate.

If we approximate the continuously DI rate by the instantaneous continuously com-

[2] See the B3 website: `http://www.b3.com.br/en_us/`.

pounding interest rate, i.e. $r_t = \ln(1+DI_t)$, the index can be represented by the following continuous compounding expression

$$y_t = y_0 \mathrm{e}^{\int_0^t r_s \mathrm{d}s}, \tag{19}$$

where r_t is given by (17).

The payoff of the IDI option maturing in t is

$$C_t = \max(y_t - K, 0), \tag{20}$$

where K is the strike price. Therefore, the price of this option is

$$C_0 \quad = \quad \mathbb{E}\left[\mathrm{e}^{-\int_0^t r_s \mathrm{d}s} \max(y_t - K, 0) \middle| \mathscr{F}_0\right]. \tag{21}$$

On the other hand, the payoff of a digital option is $\mathbb{1}_{\{y_t > K\}}$, which means that the holder receives \$ 1 at maturity if $y_t > K$ and zero otherwise. Thus, the price of a digital option is given by

$$C_0 \quad = \quad \mathbb{E}\left[\mathrm{e}^{-\int_0^t r_s \mathrm{d}s} \mathbb{1}_{\{y_t > K\}} \middle| \mathscr{F}_0\right]. \tag{22}$$

3.1 Characteristic function of integrated short-term rate

In order to price the IDI option we need to compute the characteristic function of the integral of the interest rate. Equation (21) can take the following form

$$\begin{aligned} C_0 &= \mathbb{E}\left[\mathrm{e}^{-\int_0^t r_s \mathrm{d}s} \max\left(y_0 \mathrm{e}^{\int_0^t r_s \mathrm{d}s} - K, 0\right) \middle| \mathscr{F}_0\right] \\ &= \mathbb{E}\left[\max\left(y_0 - K\mathrm{e}^{-\int_0^t r_s \mathrm{d}s}, 0\right) \middle| \mathscr{F}_0\right]. \end{aligned} \tag{23}$$

From now on we constantly benefit from the procedure found in Duffie and Singleton (2003) to obtain characteristic functions of affine jump-diffusion (AJD) models. AJD is a special type of jump diffusion models in which the drift and the variance are affine functions of the state vector. In the one-dimensional case, this means that $\mu(r_t,t) = \kappa(\theta - r_t)$ and $\sigma(r_t,t) = \sqrt{\sigma^2 + \sigma_1^2 r_t}$. The peculiar finding here is defining a (path-dependent) function of the solution of the AJD model from which a characteristic function should be obtained. In the problem we are dealing with - the IDI options under AJD models - our path-dependent function corresponds to the integral of the interest rate process, from which the closed-form expression for the associated characteristic function is calculated.

We may apply the COS method to a variety of interest rate models which have explicit characteristic functions associated with the integrated process, ranging from jump process to stochastic volatility models. Below, we supply the paper with characteristic functions and the respective cumulants of the Vasicek diffusion model and the augmented Vasicek model with exponential and normal jump amplitudes in a compound Poisson process with constant intensity.

The Vasicek model belongs to the class of Gaussian models which have constant volatility, that is $\sigma_1 = 0$. This specification provides easy calculation of bond and vanilla option prices. Moreover, Gaussian models present good forecasting performance and fitting of the yield curve (see, for instance, Dai and Singleton (2002) and Duffee (2002)). On the other hand, Gaussian models yield negative interest rates with positive probability. However, as pointed out by Duffie (2001), allowing negative rates is not necessarily wrong but only undesirable since Gaussian models can assign very low probability to negative rates with appropriated choices of the parameters.

Theorem 1 *The characteristic function associated to the integrated process $x_t = \int_0^t r_s ds$ where r_s is given by an Gaussian affine jump-diffusion model of the form* (17) *is*

$$\hat{f}(u, x_0) = \mathbb{E}\left[e^{iux_t} | x_0\right] = e^{\alpha(t) + \beta(t) r_0}, \tag{24}$$

where

$$\alpha'(s) = \beta(s)\theta\kappa + \frac{1}{2}\sigma^2\beta(s)^2 + \lambda\left[\mathbb{E}\left(e^{\beta(s)Z}\right) - 1\right], \tag{25}$$

$$\beta'(s) = -\kappa\beta(s) + iu, \tag{26}$$

with boundary conditions $\alpha(0) = 0$ and $\beta(0) = 0$.[3]

Proof. See Duffie and Singleton (2003). ■

Corollary 1 *(Characteristic function of Vasicek model) Let $\mu(r_t, t) = \kappa(\theta - r_t)$, $\sigma(r_t, t) = \sigma$ in* (17)*. The resulting process is the Vasicek model (Vasicek, 1977) for which the particular solution of* (24) *is given by*

$$\alpha(t) = -\left(\theta + \frac{iu\sigma^2}{2\kappa^2}\right)(\beta(t) - iut) - \frac{\sigma^2}{4\kappa}\beta^2(t), \tag{27}$$

$$\beta(t) = -\left(\frac{iu}{\kappa}\right)(e^{-\kappa t} - 1). \tag{28}$$

Proof. The result immediately follows from (25) with $\lambda = 0$. ■

[3] Of course, α and β are also functions of u. We omit this dependence in order to simplify the notation.

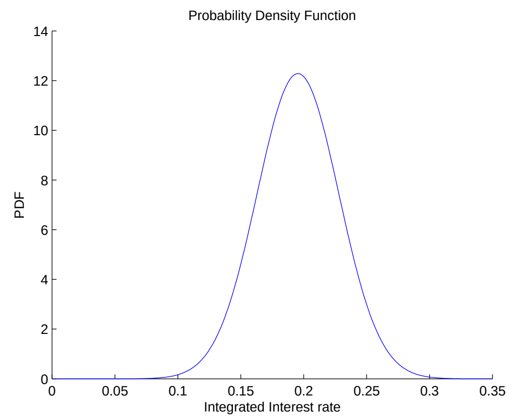

Figure 1: PDF of the Vasicek model

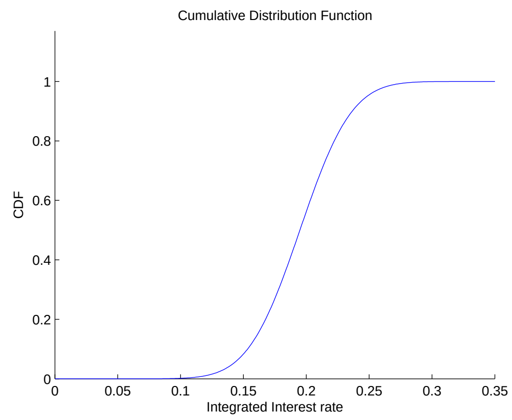

Figure 2: CDF of the Vasicek model

Corollary 2 *The cumulants of the Vasicek model are given by*

$$c_1 = \frac{\mathrm{e}^{-\kappa t}\left(\theta\kappa t\mathrm{e}^{\kappa t} + r_0\mathrm{e}^{\kappa t} - \theta\left(\mathrm{e}^{\kappa t}-1\right) - r_0\right)}{\kappa}, \tag{29}$$

$$c_2 = \frac{\sigma^2\mathrm{e}^{-2\kappa t}\left(\left(2\kappa t-3\right)\mathrm{e}^{2\kappa t} + 4\mathrm{e}^{\kappa t} - 1\right)}{2\kappa^3}, \tag{30}$$

$$c_4 = 0. \tag{31}$$

Proof. Substituting the characteristic function (24) in (16) with α and β given respectively by (27) and (28), we obtain Corollary (2). ■

Figures 1 and 2 show the probability density and the cumulative function of the integrated process approximated using (11), where r_t is given by the Vasicek process with $\kappa = 0.1265$, $\theta = 0.0802$ and $\sigma = 0.0218$.[4] We also set $r_0 = 0.1$ and $t = 2$.

Corollary 3 *(Characteristic function of Vasicek model with exponentially distributed jumps) Let $\mu(r_t,t) = \kappa(\theta - r_t)$, $\sigma(r_t,t) = \sigma$ and let the jump size Z be exponentially distributed with constant intensity λ and expected amplitude $\pm\eta$, that is, conditional on the occurrence of a jump, the density function of Z is*

$$p(z;\eta) = \frac{1}{\eta}e^{-\frac{z}{\eta}} \quad \forall\ z \geq 0. \tag{32}$$

Then the particular solution of (24) *is given by*

$$\alpha(t) = -\left(\theta + \frac{iu\sigma^2}{2\kappa^2}\right)\left(\beta(t) - iut\right) - \frac{\sigma^2}{4\kappa}\beta^2(t) - \lambda t + \frac{\lambda}{1+iu\eta}\ln\left(\frac{1\mp\beta(t)\eta}{\mathrm{e}^{-\kappa t}}\right) \tag{33}$$

$$\beta(t) = -\left(\frac{iu}{\kappa}\right)\left(\mathrm{e}^{-\kappa t}-1\right). \tag{34}$$

[4]This set of parameters are around typical values found in works that estimated Gaussian models with Brazilian data (see, for instance, da Silva et al. (2016)).

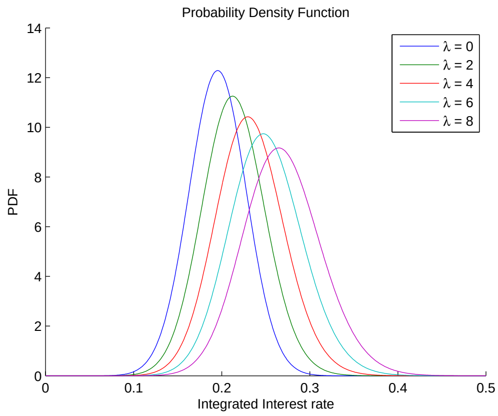

Figure 3: PDFs of the integrated short-term rate process under the augmented Vasicek model with exponentially distributed jump sizes for some jump intensities

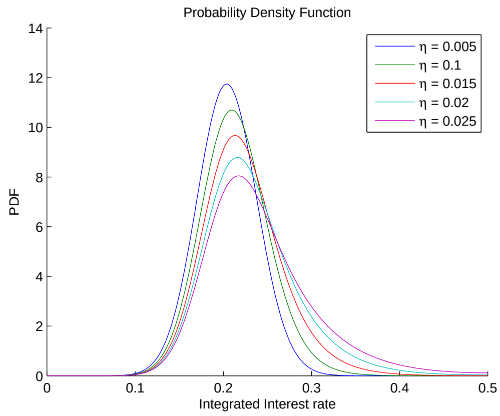

Figure 4: PDFs of the integrated short-term rate process under the augmented Vasicek model with exponentially distributed jump sizes for some jump amplitudes

Proof. Taking Z as exponentially distributed in equation (25) according to (32), we obtain Corollary 3. ■

Corollary 4 *The cumulants of the Vasicek model with exponentially distributed jumps are given by*

$$c_1 = \frac{\lambda\eta\,(\kappa t)}{(\kappa)^2} - \left(\frac{(e^{-\kappa t}-1)}{\kappa} + t\right)\theta + \frac{\lambda\eta\,(e^{-\kappa t}-1)}{\kappa^2} + \frac{r_0\,(e^{-\kappa t}-1)}{\kappa}, \tag{35}$$

$$c_2 = \frac{2\lambda\eta^2\kappa t}{\kappa^3} - \frac{\sigma^2}{k^2}\left(\frac{(e^{-\kappa t}-1)}{\kappa} + t\right) + \frac{2\lambda\eta^2\,(e^{-\kappa t}-1)}{\kappa^3} - \frac{\lambda\eta^2\,(e^{-\kappa t}-1)^2}{\kappa^3} - \frac{\sigma^2\,(e^{-\kappa t}-1)^2}{2\kappa^3}, \tag{36}$$

$$c_4 = \frac{24\lambda\eta^4\kappa t}{\kappa^5} + \frac{24\lambda\eta^4\,(e^{-\kappa t}-1)}{\kappa^5} - \frac{12\lambda\eta^4\,(e^{-\kappa t}-1)^2}{\kappa^5} + \frac{8\lambda\eta^4\,(e^{-\kappa t}-1)^3}{\kappa^5} - \frac{6\lambda\eta^4\,(e^{-\kappa t}-1)^4}{\kappa^5}. \tag{37}$$

Proof. Substituting the characteristic function (24) in (16) with α and β given respectively by (33) and (34), we obtain Corollary 4. ■

Figures 3 and 4 show the probability density functions of the integrated short-term rate process approximated using (11), where r_t is given by the Vasicek model with exponentially distributed jumps. The diffusion parameters are $\kappa = 0.1265$, $\theta = 0.0802$ and $\sigma = 0.0218$. We also set $r_0 = 0.1$ and $t = 2$. In Figure 3 we fix $\eta = 0.005$ and vary the jump intensities. On the other hand, in Figure 4 we fix $\lambda = 1$ and vary the jump amplitudes.

Corollary 5 *(Characteritic function of Vasicek model with normally distributed jumps) Let $\mu(r_t,t) = \kappa(\theta - r_t)$, $\sigma(r_t,t) = \sigma$. Assume that the jump size Z is normally distributed* *with constant intensity λ, mean m and variance Σ^2, that is, the density function of Z conditional on the occurrence of a jump is*

$$p(z;m,\Sigma) = \frac{1}{\Sigma\sqrt{2\pi}} e^{-\frac{(z-m)^2}{2\Sigma^2}} \quad \forall\ z \in \mathbb{R}. \tag{38}$$

Then the particular solution of (24) *is given by*

$$\alpha(t) = -\left(\theta + \frac{iu\sigma^2}{2\kappa^2}\right)(\beta(t) - iut) - \frac{\sigma^2}{4\kappa}\beta^2(t) - \lambda t + \lambda \int_0^t e^{\beta(l)m + \frac{(\beta(l)\Sigma)^2}{2}} dl \tag{39}$$

$$\beta(t) = -\left(\frac{iu}{\kappa}\right)(e^{-\kappa t} - 1). \tag{40}$$

Proof. The expectation in (25) is calculated according to (38). The remaining derivation follows the Corollary 1. Note that there is no closed-formula to the jump part of (39). ■

Corollary 6 *The cumulants of the Vasicek model with normally distributed jumps are given by*

$$c_1 = \frac{\mathrm{e}^{-\kappa t}\left(\theta\kappa t\mathrm{e}^{\kappa t} + r_0\mathrm{e}^{\kappa t} - \theta\left(\mathrm{e}^{\kappa t} - 1\right) - r_0\right)}{\kappa} + \frac{\lambda\mathrm{e}^{-2\kappa t}\left(\left(2\kappa m - 2\kappa^2 mt\right)\mathrm{e}^{2\kappa t} - 2\kappa m\mathrm{e}^{\kappa t}\right)}{2\kappa^3}, \tag{41}$$

$$c_2 = \frac{\sigma^2\mathrm{e}^{-2kt}\left((2kt - 3)\mathrm{e}^{2kt} + 4\mathrm{e}^{kt} - 1\right)}{2k^3} + \frac{\lambda\mathrm{e}^{-2\kappa t}\left((2\kappa t - 3)\mathrm{e}^{2\kappa t} + 4\mathrm{e}^{\kappa t} - 1\right)\left(\Sigma^2 + m^2\right)}{2\kappa^3}, \tag{42}$$

$$c_4 = 0. \tag{43}$$

Proof. Substituting the characteristic function (24) in (16) with $\alpha(t)$ and $\beta(t)$ given respectively by (39) and (40), we obtain Corollary (6). ■

The probability density functions of the integrated process approximated by (11) is shown in Figures 5 and 6. In this example, r_t is given by the Vasicek model with normally distributed jump. The diffusion parameters are $\kappa = 0.1265$, $\theta = 0.0802$ and $\sigma = 0.0218$. We also set $r_0 = 0.1$ and $t = 2$. In Figure 5 we vary the jump intensities for fixed $m = 0$ and $\Sigma = 0.02$. In Figure 6 we vary the standard deviation Σ with a fixed jump intensity ($\lambda = 2$).

4 IDI option pricing with the COS method

In the previous section, we obtain the characteristic function of the random variable $\int_t^T r_s ds$ which enters in A_j coefficients in Equation (13). Therefore, in order to price the IDI options, we have to calculate the corresponding B_j coefficients. In this section,

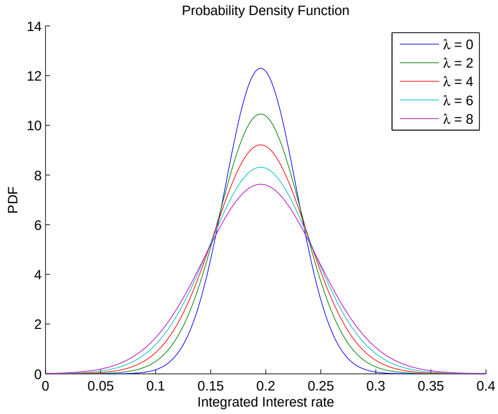

Figure 5: PDF of the integrated short-term rate process under the augmented Vasicek model with normally distributed jump sizes for some jump intensities

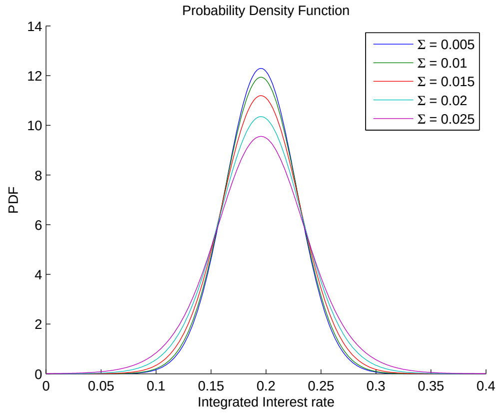

Figure 6: PDF of the integrated Vasicek model process under the augmented Vasicek model with normally distributed jump sizes for some standard-deviation of jump size

we consider the vanilla call option case as shown in Equation (20) and the digital option defined in Equation (22).

4.1 B_j coefficients for vanilla options

Theorem 2 *The B_j coefficients for vanilla IDI call options are given by*

$$B_0 = \int_{-\ln\left(\frac{y_0}{K}\right)}^{b} y_0 - K\mathrm{e}^{-x}\mathrm{d}x = y_0\left(\ln\left(\frac{y_0}{K}\right) + b - 1\right) + \mathrm{e}^{-b}K, \tag{44}$$

and

$$B_j = \int_{-\ln\left(\frac{y_0}{K}\right)}^{b} \left(y_0 - K\mathrm{e}^{-x}\right)\cos\left(\frac{\pi j\,(x-a)}{b-a}\right)\mathrm{d}x =$$
$$\frac{(b-a)\,\mathrm{e}^{-b}\left(\left(b^2-2ab+a^2\right)\mathrm{e}^{b}y_0\sin\left(\frac{\pi j\ln\left(\frac{y_0}{K}\right)+\pi a j}{b-a}\right) + (\pi a-\pi b)\,\mathrm{e}^{b}\,j y_0\cos\left(\frac{\pi j\ln\left(\frac{y_0}{K}\right)+\pi a j}{b-a}\right)\right)}{\pi j\,(\pi^2 j^2+b^2-2ab+a^2)} +$$
$$\frac{(b-a)\,\mathrm{e}^{-b}\left(\left(\pi^2\mathrm{e}^{b}j^2+\left(b^2-2ab+a^2\right)\mathrm{e}^{b}\right)\sin(\pi j)\,y_0 + \left((\pi b-\pi a)\,j\cos(\pi j) - \pi^2 j^2\sin(\pi j)\right)K\right)}{\pi j\,(\pi^2 j^2+b^2-2ab+a^2)}. \tag{45}$$

Proof. The vanilla IDI call option payoff is given by (20). Integrating it according to equation (14) gives Theorem 2. ■

4.2 B_j coefficients for digital options

Theorem 3 *The B_j coefficients for digital IDI call options are given by*

$$B_0 = \int_{-\ln\left(\frac{y_0}{K}\right)}^{b} \mathrm{d}x = \ln\left(\frac{y_0}{K}\right) + b, \tag{46}$$

and

$$\begin{aligned} B_j &= \int_{-\ln\left(\frac{y_0}{K}\right)}^{b} \cos\left(\frac{\pi j (x-a)}{b-a}\right) \mathrm{d}x \\ &= \frac{(b-a)}{\pi j} \left(\sin\left(\frac{\pi j \left(\ln\left(\frac{y_0}{K}\right) + a\right)}{b-a}\right) + \sin(\pi j) \right). \end{aligned} \tag{47}$$

Proof. The digital IDI call option payoff is given by (22). Integrating it according to equation (14) gives Theorem 3. ■

5 Hedging via Fourier series

Equation (13) is suitable to the straight developing of the hedging parameters. As only the B_j terms include information about the *IDI*, the delta hedging is simply given by

$$\frac{\partial C(0,t)}{\partial y_0} = \sum_{k=0}^{n} A_j \frac{\partial B_j}{\partial y_0}. \tag{48}$$

5.1 European vanilla options

Theorem 4 *The Δ_j coefficients for vanilla IDI call options are given by*

$$\Delta_0 = \frac{\partial B_0}{\partial y_0} = \ln\left(\frac{y_0}{K}\right) + b, \tag{49}$$

and for $j \geq 1$

$$\Delta_j = \frac{\partial B_j}{\partial y_0} = \frac{(b-a)}{\pi j} \left(\sin\left(\frac{\pi j \ln\left(\frac{y_0}{K}\right) + \pi a j}{b-a}\right) + \sin(\pi j) \right). \tag{50}$$

Proof. The Theorem follows directly from (44) and (45). ■

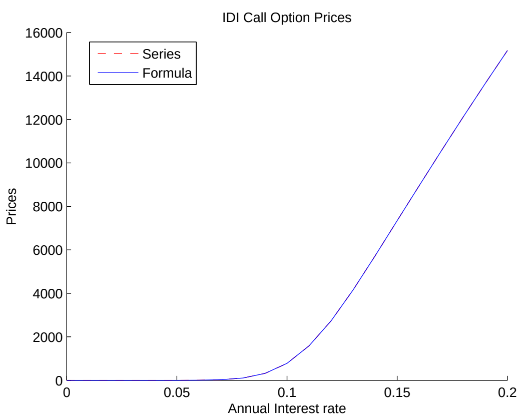

Figure 7: Vanilla IDI option price under the Vasicek model

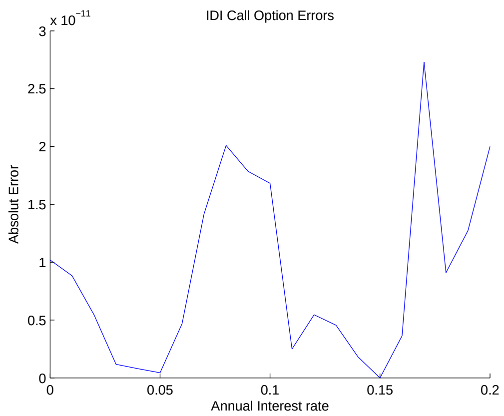

Figure 8: Absolute error of the vanilla IDI option price under the Vasicek model

5.2 European digital options

Theorem 5 *The* Δ_j *coefficients for digital IDI call options are given by*

$$\Delta_0 = \frac{\partial B_0}{\partial y_0} = \frac{1}{y_0}, \tag{51}$$

and for $j \geq 1$

$$\Delta_j = \frac{\partial B_j}{\partial y_0} = \frac{\cos\left(\frac{\pi j\left(\ln\left(\frac{y_0}{K}\right)+a\right)}{b-a}\right)}{y}. \tag{52}$$

Proof. The Theorem follows directly from (46) and (47). ■

6 Computational analysis and numerical results

In this section we analyzed the IDI option prices given by the COS method. First, we assume that the short rate evolves according to the Vasicek model. In this simple case, Vieira and Pereira (2000) provide a closed-form solution to the IDI call price which allow us to investigate the error of the COS method. In this exercise, we work with the same values of parameters used in da Silva et al. (2016), namely, $\kappa = 0.1265$, $\theta = 0.0802$ and $\sigma = 0.0218$. The time to maturity of the option is two years and its strike price is 123,000. The IDI spot (IDI_0) is 100,000. Figure 7 shows the prices of this option as function of r_0 calculated by the COS method and by the closed-form developed by Vieira and Pereira (2000). Note that the prices are visually indistinguishable which is highlighted in Figure 8, where we can see that the error is of the order of 10^{-11} for $n = 100$.

In order to investigate the convergence of the COS method, we analyze the pricing and hedging errors of the IDI call options when we increase n exponentially, that is, $n = 2^p$. The benchmark is the price of the option provide by the analytical result shown in Vieira and Pereira (2000). We use the cumulants given in (29) and (30) to determine the integration limits calculated via (15). The upper panel of Table 1 shows the convergence results for the price while the lower panel presents the converge results for the delta-hedging. We highlight the fast convergence of the prices and the deltas. Moreover, note that increases in n add a little computational effort.[5]

Table 1: Convergence Analysis

Pricing					
p	3	4	5	6	7
Time (sec)	0.0151	0.0153	0.0158	0.0159	0.0159
Abs. Err.	81.75×10^0	1.43×10^0	1.18×10^{-5}	2.53×10^{-11}	2.52×10^{-11}

Hedging					
p	3	4	5	6	7
Times (sec)	0.0233	0.024	0.024	0.025	0.025
Abs. Err.	0.02×10^0	6.84×10^{-4}	1.13×10^{-8}	1.99×10^{-15}	1.99×10^{-15}

We also analyze the convergence varying the coefficient L of equations (15) and fixed $p = 7$. Figure 9 presents the results. In the left panel we show the fast convergence for a very short maturity of 2 days of an at-the-money call option (strike price equals 100,100). In the right panel we repeat the process for a very long maturity of 10 years (strike price equals 271,828). In both cases the method meets satisfactory error for $L \geq 6$. The results are very similar to those presented in Fang and Oosterlee (2008) for options on stocks.

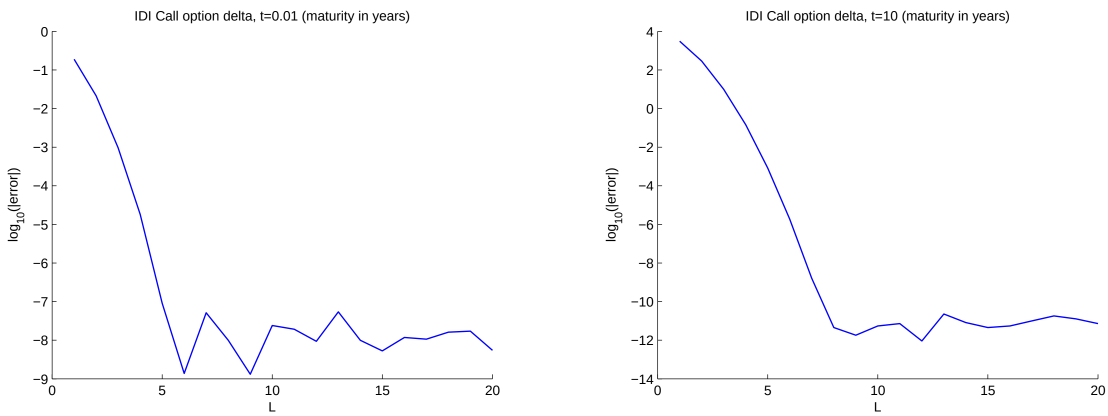

Figure 9: Error convergence as a function of L.

[5]The computer used for all experiments has an Intel Core i5 CPU, 2.53GHz. The code was written in MATLAB 7.8.

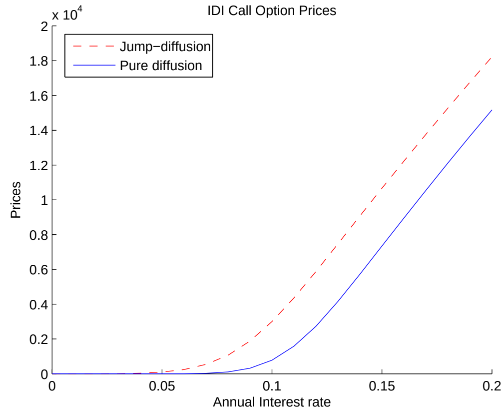

Figure 10: Vanilla IDI option price under the augmented Vasicek model with exponentially distributed jump sizes (positive jumps)

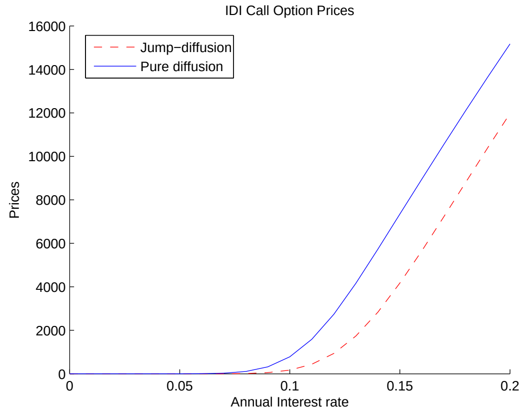

Figure 11: Vanilla IDI option price under the augmented Vasicek model with exponentially distributed jump sizes (negative jumps)

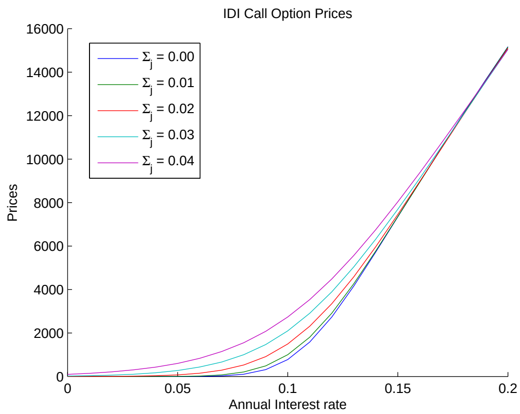

Figure 12: Vanilla IDI option price under the augmented Vasicek model with normally distributed jump sizes for some standard-deviation of the jump amplitude

To study the IDI option price behavior due to discontinuities in the interest rate, we add a jump process to the pure Vasicek model. In this case, we do not have a close-form solution to the IDI option price. Therefore, we only investigate as the call prices varies when we change the model jumps parameters. First, we consider an exponential jump component with $\lambda = 4$ and $\eta = \pm 0.005$. The diffusion parameters are the same of the previous exercises. Figure 10 presents the prices as a function of r_0 assuming a positive jump. In Figure 11 we show the price for a negative η. As expected, a positive jump raises the price of the option, while a negative decreases it. Next, we enhance the basic model with normal jumps. Figure 12 shows the call prices for some values of Σ with fixed $\lambda = 2$ and $m = 0$. A normal jump increases the price of the IDI option. The higher the volatility of the jump, the higher the price. This is expected finding since the normal jump is a source of risk, therefore must be pricing.

To further demonstrate the powerful of the COS method, we analyze the pricing

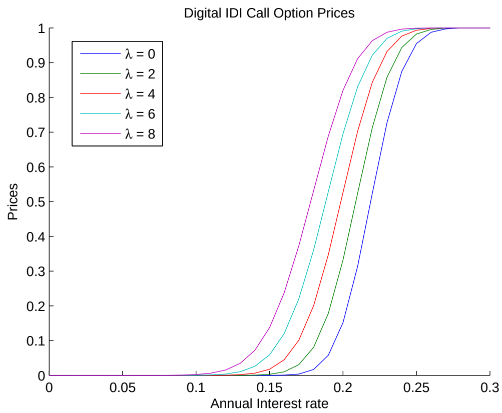

Figure 13: Digital IDI option price under the augmented Vasicek model with exponentially distributed jump sizes for some jump intensities

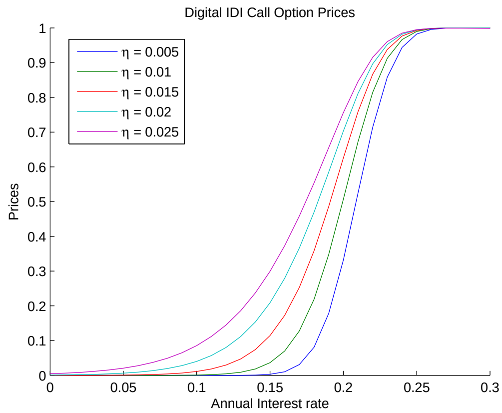

Figure 14: Digital IDI option price under the augmented Vasicek model with exponentially distributed jump sizes for some jump amplitudes

problem of an exotic option on the IDI. Specifically, we price the digital IDI call option (see Equation 22). We again use the diffusion parameters aforementioned adding exponential jump components. In Figure 13 we show the price as function of λ while in Figure 14 we vary the η parameter. For these digital IDI options examples we use a strike price equals 150,000 with the other features of the call kept unchanged. Note that the higher λ or η, the higher the option price. This result is in accordance with the finance theory, since both the intensity and the amplitude of the jump are sources of risk.

As final exercise, we show in Figure 15 a possible volatility skew implied by the jump-diffusion models. The parameters for the Vasicek model with exponentially distributed jump process are $\kappa = 1$, $\theta = 0.05$, $\sigma = 0.03$, $\lambda = 5$ and $\eta = 0.0025$. We also set $r_0 = 0.10$ and $t = 1$. After computing the options prices for different strikes, we follow the market practice, obtaining the implied volatilities by the Black (1976) model. Note that the model is able to reproduce the stylized fact of skew volatility: the higher the moneyness, the higher the Black implied volatility.

7 Conclusion

We extended the range of application of the COS method to interest rate derivatives contracts. We obtain the prices and the hedging parameters for a special financial product found in the Brazilian Market, the IDI option. The method allows fast simultaneous calibration for different models, which have characteristic function in closed-form, since the payoff is treated independently of the stochastic factors. We also provide a variety of characteristic functions of affine jump-diffusion integrated process, which is used to calculate the price of such product. The numerical results demonstrate the effectiveness of

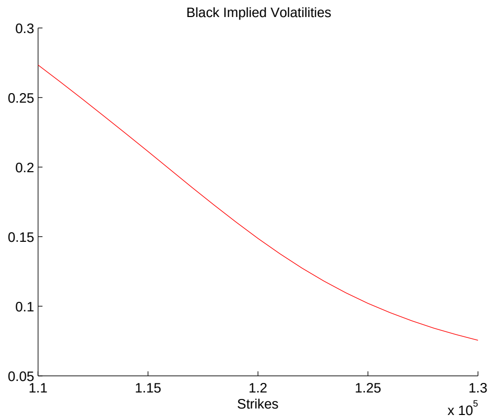

Figure 15: Volatility skew

the COS method for interest rate derivatives in a similar way that it was already reported for stock options in other works. It is important to emphasize that due to the nature of the COS method, it is easily implemented on parallel computers, speeding up the convergence results.

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