ISSN 1518-3548

BANCO CENTRAL DO BRASIL

Working Paper Series

8

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José Alvaro Rodrigues Neto

September, 2000

ISSN 1518-3548

CGC 00.038.166/0001-05

Working Papers Series	Brasília	n. 8	Oct	2000	P. 1 – 19

Working Paper Series

Edited by:

Research Department (Depep)
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The Correlation Matrix of the Brazilian Central Bank's Standard Model for Interest Rate Market Risk

José Alvaro Rodrigues Neto[*]

Abstract

Central Bank of Brazil is implementing a Value At Risk (V.A.R.) methodology to establish minimum capital requirements for financial institutions to bear market risk derived from interest rate fluctuations. This article shows that the construction of the correlation matrix of the Brazilian Central Bank's Standard Model for Interest Rate is coherent, in the sense it is positive defined.

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[*] Central Bank of Brazil. The author thanks Alexandre Tombini, André Amante, Fabio Araujo, Guilherme Arcoverde, Marta Baltar Moreira and Sérgio Werlang for their help. The remaining errors are only my responsability.

The Correlation Matrix of the Brazilian Central Bank's Standard Model for Interest Rate Market Risk

1. Introduction

Brazilian Central Bank recently established capital requirements in order to prevent market risk derived from interest rate fluctuations. It was constructed a model to do so, named the Standard Model for Interest Rate Market Risk. It is based on a value at risk (VaR) methodology.

This article deals with a technical aspect of Brazilian Central Bank's Standard Model for capital requirements for financial institutions to bear market risk. This Standard Model has a parameterized structure. Within this structure, the correlation matrix of the series of the factors associated to the so called vertexes of the term structure of the interest rate has a crucial importance.

This concepts were implemented by the construction of a matrix depending on two parameters that captures the observed behavior of the historical correlation between the vertexes.

The problem that the construction brought was to guarantee that for all range of the parameters the adopted procedures are mathematically correct, that is, the correlation matrix must be positive defined, so the calculated value at risk is a well defined real positive number.

The Brazilian VaR model for interest rate risk can be found in [1] and [2]. For a general treatment of VaR see [3] or the references there.

This work is divided in four sections, one table and 4 graphs. Section 2 explains the mathematical problem. Section 3 shows two ideas used to attack the problem, while section 4 explains the nature of the difficulty in solving it analytically, argues why it is necessary to use careful numerical procedures and explains why it is safe to say that the correlation matrix is in fact positive defined for a certain range of the parameters.

2. The Mathematical Problem

The 7x7 matrix of correlation A_{7x7} with coefficients $\left(\rho_{ij}\right)_{i,j=1}^{7}$ is symmetric, with main diagonal of ones. The problem is to prove that it is positive defined for a certain range of values of its two parameters ρ, k.

The definition of the coefficients are:

$$\rho_{ij} = \rho + (1-\rho)^{m(i,j)},$$

where $m(i,j) = (b_{ij})^k$, and the matrix $(b_{ij})_{1\le i,\, j\le 7}$ is given by:

$$(b_{ij})_{1\le i,\, j\le 7} = \begin{pmatrix} 1 & 2 & 3 & 6 & 12 & 24 & 36 \\ 2 & 1 & 1.5 & 3 & 6 & 12 & 18 \\ 3 & 1.5 & 1 & 2 & 4 & 8 & 12 \\ 6 & 3 & 2 & 1 & 2 & 4 & 6 \\ 12 & 6 & 4 & 2 & 1 & 2 & 3 \\ 24 & 12 & 8 & 4 & 2 & 1 & 1.5 \\ 36 & 18 & 12 & 6 & 3 & 1.5 & 1 \end{pmatrix}$$

with the parameters $\rho,\ k$ varying between $0.1 \le \rho \le 0.9$ and $0.1 \le k \le 0.9$.

Let $K = [0.1, 0.9] \times [0.1, 0.9]$. It is clearly a compact set.

It is easy to see that A_{7x7} is symmetric. It is a well known fact that every symmetric matrix is equivalent to a diagonal matrix of its eigenvalues, which are real numbers. So it will be positive defined if and only if all eigenvalues (and hence the minimum of them) are strict positive numbers.

So it is enough to consider the function $f : K \subset \Re^2 \to \Re$ that takes the parameters ρ, k in the minimum of the eigenvalues. Nevertheless it is not easy to measure the sensibility of $f : K \subset \Re^2 \to \Re$ with respect to ρ, k, that is, to calculate $\frac{\partial f}{\partial \rho}, \frac{\partial f}{\partial k}$. In fact, it is not clear that this function is differentiable or even if it is continuos[1]. The figures 1 and 2 below show that when a parameter of a polynomial is changing smoothly, the minimum real zero of the polynomial can change in a non smooth way:

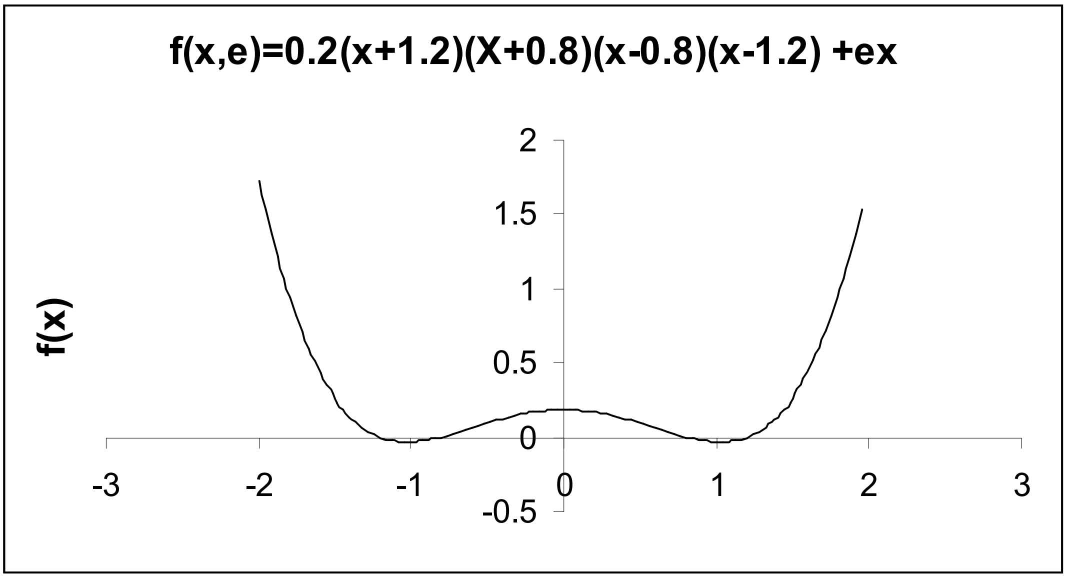

Figure 1: $f(x,0) = 0$ has 4 real roots. The lower one is smaller than -1.

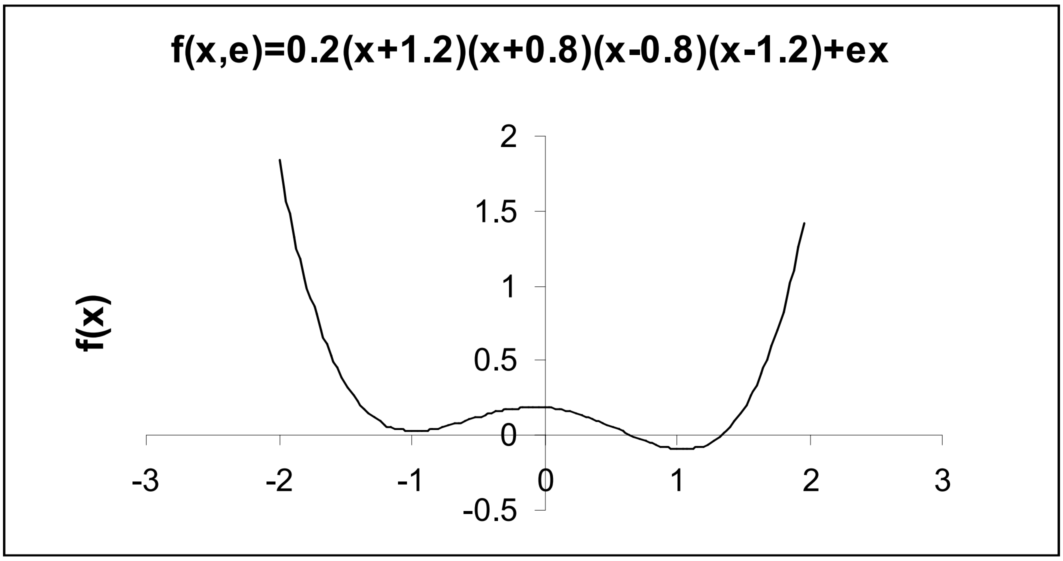

Figure 2: $f(x,-0.06)=0$ has only two real roots. The smaller one is greater than zero.

Figures 3 and 4 in the end show the complete graph of $f : K \subset \Re^2 \to \Re$ and its level curves. Figures 5 and 6 show zooms at particular region of the graph, where f has a strange behavior. Notice that f is always a continuos function.

[1] In this case *f* will be continuos.

3. The Solution

3.1 First Solution

Initially the idea was to use the following well known theorem:

Theorem (Sylvester's Criterion for a Positive Defined Matrix):

Let A be a real symmetric square matrix. So A is positive defined if and only if $\det A > 0$ and $\det A_k > 0$ for all minors A_k, the square sub matrix of A, of order k, obtained from A by deleting the last rows and columns, that is:

$$\rho_{11} > 0\,;\ \begin{vmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{vmatrix} > 0\,;\ \ldots;\ |A| > 0\,.$$

The algebraic expressions for this minors have many parcels and most of them are non analytical, so whether all the minors are positive in the desirable range of the parameters ($0.1 \le \rho \le 0.9$ and $0.1 \le k \le 0.9$, a continuos range) can only be verified numerically.

A serious problem is that it is necessary to make an infinity number of calculations to check it out.

3.2 Second Approach – Spectral Decomposition

An different approach could be given by the observation that it is enough to prove that for all $x \in \Re^7$ the quadratic form Q defined by:

$$Q : \Re^7 x \Re^7 \to \Re$$

$$Q(x, y) = \sum_{i=1}^{7} \sum_{j=1}^{7} \rho_{i,j} x_i y_j$$

and associated with the symmetric matrix A, with coefficients ρ_{ij}, has the property:

$Q(x, x) = x' \cdot A \cdot x \geq 0$, $\forall x \in \Re^7$, with equality holding if and only if $x = 0 \in \Re^7$.

The main idea was the study of the problematic directions, that is, the directions where $Q(x, x)$ assumes its minimum values. Formally, since $Q(x, y)$ is a homogeneous function of second degree, it is enough to prove the claim for all $x \in S^6$, the six dimensional sphere with unitary radius and center in the origin of $\Re^7$.

The Signal Matrix

In order to find which parcels of $Q(x, x)$ are positive it can be defined a 7x7 matrix Σ, whose coefficients are either the positive sign (+) or the negative sign (-).

For instance, if x_j is positive for $j=1,2,3,4$ and negative for $j=5,6,7$ the matrix becomes:

$$\Sigma=\begin{bmatrix} + & + & + & + & - & - & - \\ + & + & + & + & - & - & - \\ + & + & + & + & - & - & - \\ + & + & + & + & - & - & - \\ - & - & - & - & + & + & + \\ - & - & - & - & + & + & + \\ - & - & - & - & + & + & + \end{bmatrix}$$

It can be observed that the problematic directions are those corresponding to the vectors $x \in S^6$ with 4 positive coordinates and 3 negative coordinates or vice versa. In this case it can be easily checked there are 25 positive parcels and 24 negative ones in $Q(x,x)$.

These are the configurations of signs from the x_j, $j=1,\cdots,7$, that gives the minimum net amount of positive signs. So, without loss of generality, it can be supposed there are 4 positive values of x_j and 3 negative ones.

Let $x_k, x_l, \mathrm{x}_{\mathrm{m}}, x_n$ be positive and $x_p, x_q, \mathrm{x}_{\mathrm{r}}$ be negative, with:

$$\{1,\cdots,7\}=\{k,l,m,n,p,q,r\}$$

In this case:

$$Q(x,x)=\sum_{i=1}^{7}\rho_{ii}x_i^2+2\sum_{i,j}^{9\oplus}\rho_{ij}x_ix_j+2\sum_{i,j}^{12-}\rho_{ij}x_ix_j$$

where $\sum_{i,j}^{9\oplus}$ represents positive parcels, that is, (i,j) such that $i,j \in \{k,l,m,n\}$ or $i,j \in \{p,q,r\}$ and $\sum_{i,j}^{12-}$ represents the negative ones ($i \in \{k,l,m,n\}, j \in \{p,q,r\}$ or vice versa).

It was numerically verified that the parameters $\rho = 0.1$, $k = 0.1$ of A are those which make it have the lower minimum eigenvalue. With this parameters it can be observed that all the coefficients ρ_{ij} are about the same, say $\overline{\rho}$, between 0.96 and 0.99. A table with some matrixes for different parameters is presented in the end.

Fix the worst direction, that is, the direction that makes the smaller eigenvalue of the matrix A (with fixed parameters $\rho = 0.1$, $k = 0.1$) assumes its minimum. This direction is formally represented by $x \in S^6$. Associated with this fixed values of x_j there is an unique signal matrix Σ.

In this case:

$$Q(x,x) \cong \overline{\rho}.R(x,x), \text{ where } R(x,x) = \sum_{i=1}^{7} x_i^2 + 2\sum_{i,j}^{9\oplus} x_i x_j + 2\sum_{i,j}^{12-} x_i x_j$$

So the problem becomes to minimize $R(x,x)$, with $x \in S^6$. But R is an semi-positive quadratic form. Its defined by a symmetric matrix (by abuse of notation it is also called R) with eigenvalues $7,0,0,0,0,0,0$. The corresponding eigenvectors forms an orthogonal basis of $\Re^7$.

Let v_7 be the eigenvector corresponding to the eigenvalue 7. It can be seen $R(x,x) = 0,\ \forall x$ such that $\langle x, v_7 \rangle = 0$. So, there is an hiper-plane H of co-dimension one, such that for all $x \in H$, $R(x,x) = 0$.

In that way $Q(x,x)$ can be extremely close to zero in an set with strict positive Lebesgue measure. Whether $Q(x,x)$ will be greater than zero depends on the combination of two factors: the chosen direction and the values ρ_{ij} (the parameters). At each fixed direction (Σ uniquely defined) the sign of $Q(x,x)$ will be determinate by the sums:

$$2\sum_{i,j}^{12-} \rho_{ij} x_i x_j \quad \text{and} \quad \sum_{i=1}^{7} \rho_{ii} x_i^2 + 2\sum_{i,j}^{9\oplus} \rho_{ij} x_i x_j \,.$$

4. The Results and Conclusions

Within the parameters ($0{,}1 \le \rho, k \le 0{,}9$) the Brazilian Central Bank adopts, the region where the lower eigenvalue of the matrix A has its minimum is a neighborhood[2] of the point:

$$\begin{cases} \rho = 0.1 \\ k = 0.1 \end{cases}$$

being this value approximately 0.0028.

[2] In fact, this point is the minimum.

For this the numerical values of the function $f:\Re^2 \rightarrow \Re$ and its partial derivatives $\frac{\partial f}{\partial \rho}, \frac{\partial f}{\partial k}$ were calculated. By the mean value theorem the desired result follows[3]. However, it should be clear that it was used a numerical procedure.

So, within the specified parameters $\rho,\ k$ of A it is possible to guarantee that A is positive defined.

Nevertheless, the pure analytical prove by Sylvester's Criterion is not possible because the determinants of all the minors A_k are made of many parcels and a great part of them are non analytic. Checking if these expressions are positive for all range of the parameters is a problem similar to find a solution for the equation $2^x = x$. It can only be done numerically.

The methodology develop in this study can be easily expanded for similar problems[4].

It is also interesting to observe the behavior of the function:

$f:[0.1,\ 0.9]\times[0.1,\ 0.9]\rightarrow \Re$ (which takes the parameters $\rho,\ k$ in the minimum eigenvalue of A) in the region:

$$\begin{cases} 0.1 \le \rho \le 0.3 \\ 0.7 \le k \le 0.9 \end{cases}$$

It looks like it is close to a non continuos point. See figure 5 in the end.

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[3] In fact, to be more rigors the numerical error of the derivatives should be calculated by the second derivatives, being their errors corrected by the third derivatives and so on. But numerical values are sufficient to ensure the desired results.

[4] Similar problems with a different formula for the coefficients or with different dimension.

References

[1] Lins Arcoverde, Guilherme “Alocação de Capital para Cobertura do Risco de Mercado de Taxas de Juros de Natureza Prefixada”, Dissertação de Mestrado, EPGE/FGV, 2000.

[2] Research Department, Central Bank of Brazil “The Brazilian Central Bank’s Model for Interest Rate Market Risk”, Working Paper Series, Banco Central do Brasil.

[3] Jorion, Philippe ”Value at Risk New Benchmark for Controlling Market Risk”, Mc Graw Hill.

Table 1

			A(i,j) =								m(i,j)			
Rho 0.1	1.00	0.99	0.99	0.98	0.97	0.97	0.96	1.00	1.07	1.12	1.20	1.28	1.37	1.43
k 0.1	0.99	1.00	1.00	0.99	0.98	0.97	0.97	1.07	1.00	1.04	1.12	1.20	1.28	1.34
minimum eigenvalue	0.99	1.00	1.00	0.99	0.99	0.98	0.97	1.12	1.04	1.00	1.07	1.15	1.23	1.28
	0.98	0.99	0.99	1.00	0.99	0.99	0.98	1.20	1.12	1.07	1.00	1.07	1.15	1.20
	0.97	0.98	0.99	0.99	1.00	0.99	0.99	1.28	1.20	1.15	1.07	1.00	1.07	1.12
	0.97	0.97	0.98	0.99	0.99	1.00	1.00	1.37	1.28	1.23	1.15	1.07	1.00	1.04
0.0028	0.96	0.97	0.97	0.98	0.99	1.00	1.00	1.43	1.34	1.28	1.20	1.12	1.04	1.00

			A(i,j) =								m(i,j)			
Rho 0.1	1.00	0.92	0.85	0.69	0.47	0.26	0.17	1.00	1.87	2.69	5.02	9.36	17.47	25.16
k 0.9	0.92	1.00	0.96	0.85	0.69	0.47	0.34	1.87	1.00	1.44	2.69	5.02	9.36	13.48
minimum eigenvalue	0.85	0.96	1.00	0.92	0.79	0.60	0.47	2.69	1.44	1.00	1.87	3.48	6.50	9.36
	0.69	0.85	0.92	1.00	0.92	0.79	0.69	5.02	2.69	1.87	1.00	1.87	3.48	5.02
	0.47	0.69	0.79	0.92	1.00	0.92	0.85	9.36	5.02	3.48	1.87	1.00	1.87	2.69
	0.26	0.47	0.60	0.79	0.92	1.00	0.96	17.47	9.36	6.50	3.48	1.87	1.00	1.44
0.0328	0.17	0.34	0.47	0.69	0.85	0.96	1.00	25.16	13.48	9.36	5.02	2.69	1.44	1.00

			A(i,j) =								m(i,j)			
Rho 0.9	1.00	0.98	0.98	0.96	0.95	0.94	0.94	1.00	1.07	1.12	1.20	1.28	1.37	1.43
k 0.1	0.98	1.00	0.99	0.98	0.96	0.95	0.95	1.07	1.00	1.04	1.12	1.20	1.28	1.34
minimum eigenvalue	0.98	0.99	1.00	0.98	0.97	0.96	0.95	1.12	1.04	1.00	1.07	1.15	1.23	1.28
	0.96	0.98	0.98	1.00	0.98	0.97	0.96	1.20	1.12	1.07	1.00	1.07	1.15	1.20
	0.95	0.96	0.97	0.98	1.00	0.98	0.98	1.28	1.20	1.15	1.07	1.00	1.07	1.12
	0.94	0.95	0.96	0.97	0.98	1.00	0.99	1.37	1.28	1.23	1.15	1.07	1.00	1.04
0.0067	0.94	0.95	0.95	0.96	0.98	0.99	1.00	1.43	1.34	1.28	1.20	1.12	1.04	1.00

			A(i,j) =								m(i,j)			
Rho 0.9	1.00	0.91	0.90	0.90	0.90	0.90	0.90	1.00	1.87	2.69	5.02	9.36	17.47	25.16
k 0.9	0.91	1.00	0.94	0.90	0.90	0.90	0.90	1.87	1.00	1.44	2.69	5.02	9.36	13.48
minimum eigenvalue	0.90	0.94	1.00	0.91	0.90	0.90	0.90	2.69	1.44	1.00	1.87	3.48	6.50	9.36
	0.90	0.90	0.91	1.00	0.91	0.90	0.90	5.02	2.69	1.87	1.00	1.87	3.48	5.02
	0.90	0.90	0.90	0.91	1.00	0.91	0.90	9.36	5.02	3.48	1.87	1.00	1.87	2.69
	0.90	0.90	0.90	0.90	0.91	1.00	0.94	17.47	9.36	6.50	3.48	1.87	1.00	1.44
0.0601	0.90	0.90	0.90	0.90	0.90	0.94	1.00	25.16	13.48	9.36	5.02	2.69	1.44	1.00

Table 1: matrixes A and m(i,j), for four sets of the parameters ρ, k and its minimum eigenvalue.

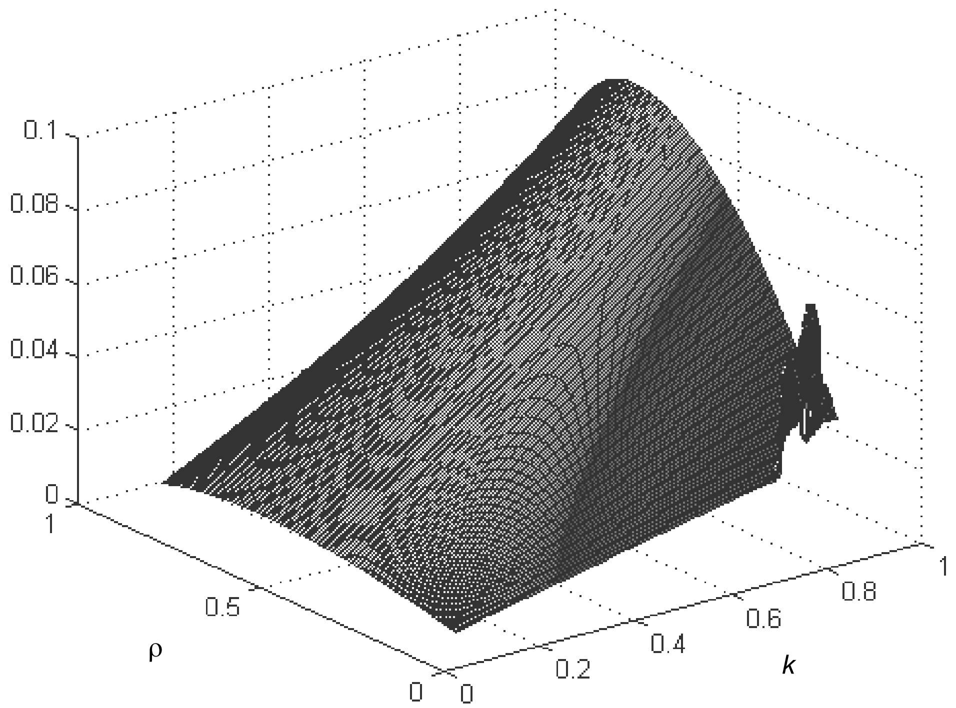

Figure 3: function $f : K \subset \Re^2 \to \Re$. The vertical axis represents the values of f. The ρ axis has only one division (at 0.5) and the k axis has marks at 0.2, 0.4, 0.6 and 0.8.

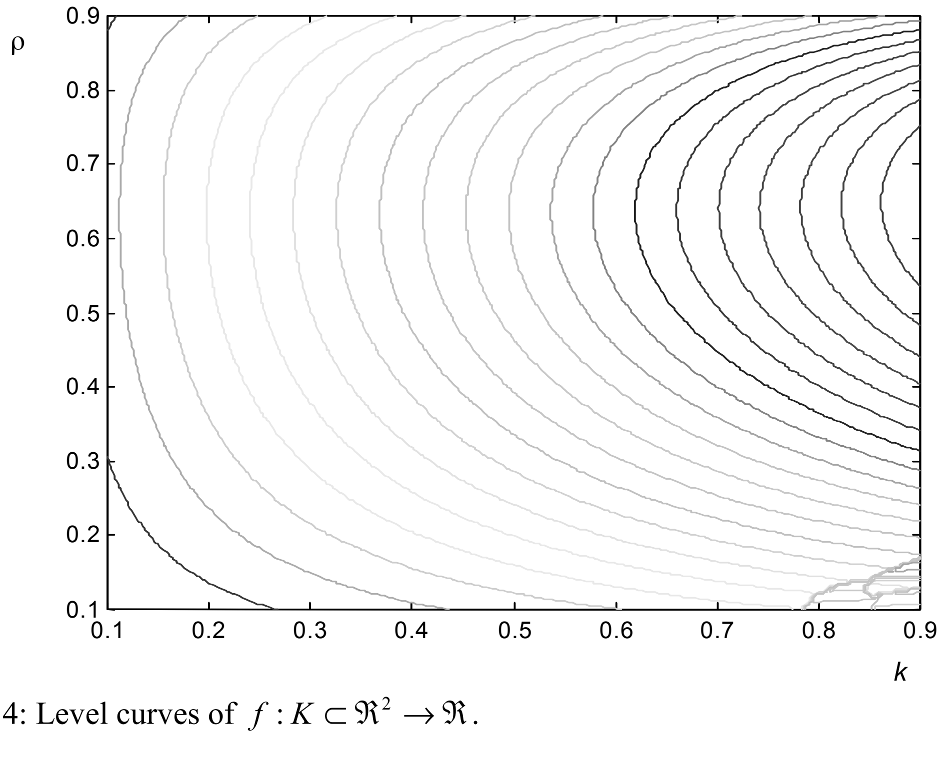

Figure 4: Level curves of $f : K \subset \Re^2 \to \Re$.

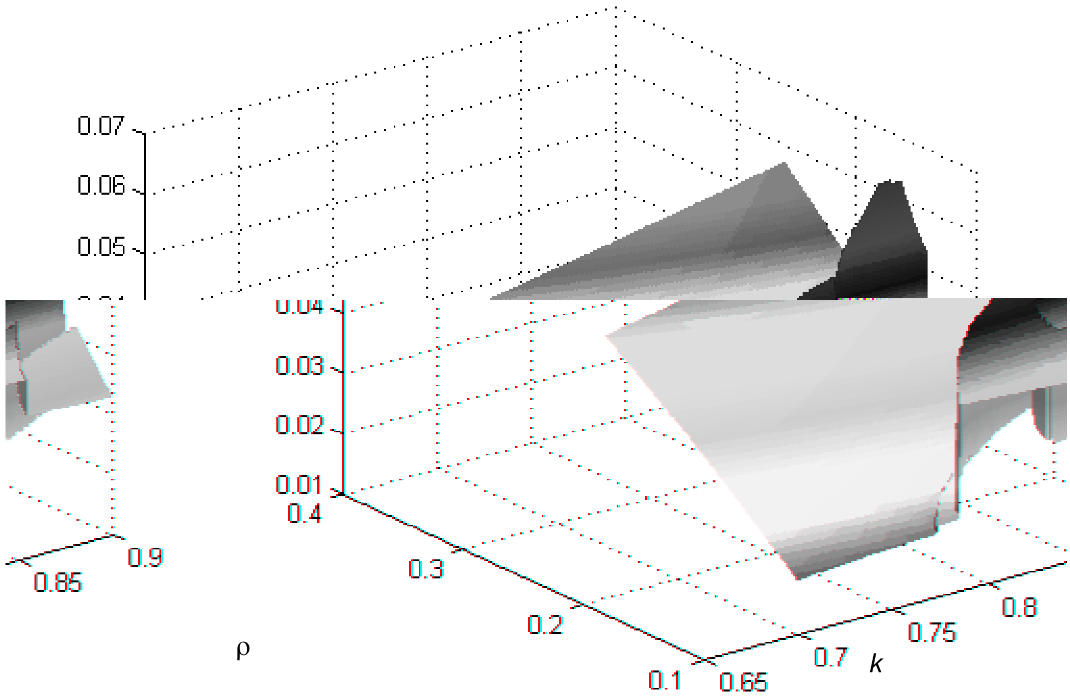

Figure 5: the “strange” region of $f : K \subset \Re^2 \to \Re$

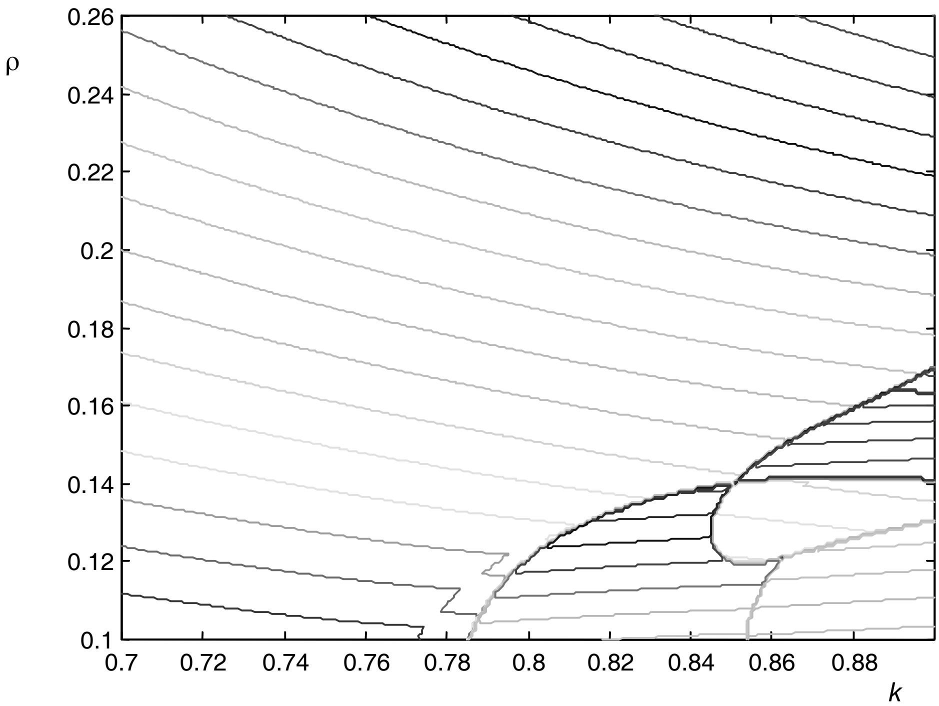

Figure 6: level curves of the previous graph.

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